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Massive Field Equations of Arbitrary Spin in Schwarzschild Geometry: Separation Induced by Spin-$\frac{3}{2}$ Case

Antonio Zecca[1]

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The separation of variables of the spin-$\frac{3}{2}$ field equation is performed in detail in the Schwarzschild geometry by means of the Newman Penrose formalism. The separated angular equations coincide with those relative to the Robertson-Walker space-time. The separated radial equations, that are much more entangled, can be reduced to four ordinary differential equations, each in one only radial function. As a consequence of the particular nature of the spin coefficients it is shown, by induction, that the massive field equations can be separated for arbitrary spin.

KEY WORDS: Schwarzschild geometry; massive field equations; variables separation; exact solutions.

PACS 04.20.Cv - Fundamental problems and general formalism.
PACS 03.65.Pm - Relativistic wave equations.
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1. INTRODUCTION

The solution of the massive field equation of arbitary spin is a problem of mathematical and physical interest, specially in concrete examples of space-time. Recently the massive field equations have been sepearated for arbitrary spin in the Robertson-Walker (R-W) space-time. The result has been obtained by extending a separation of variables method employed to integrate the Dirac equation (Zecca, 1996), the spin-1, spin-$\frac{3}{2}$ and spin-2 equations in R-W space-time (Zecca, 2005, 2006, 1996b). Another context of physical interest where to study the equations is that of the Schwarzschild space-time. In that metric wide studies have been done in case of the lower values of the spin. For the scalar field case see e.g. Boulware, 1975; Zecca, 2000 and references therein. The Dirac equation results

[1] Dipartimento di Fisica dell' Universita', Via Celoria 16, I-20133 Milano, Italy; INFN, Sezione di Milano, and GNFM, Italy; e-mail: Antonio.Zecca@mi.infn.it.

to be separable as a consequence of the pioneer paper by Chandrasekhar (1976) who separated the Dirac equation in Kerr metric. (See also Boulware, 1975a; Zecca, 1998, 2000). The spin 1 equation, in the form of Proca fields equation, was separated by Gal'tsov *et al.*, 1984 and, in the general form, independently, by Zecca (2005). Also studies of massive vector fields perturbations in Schwarzschild space-time involves variable separation (e.g. Konoplya, 2006).

However, as far as the author knows, the proof that the massive field equations of arbitrary spin are separable in the variables in Schwarzschild metric seems to lack. It is the object of the present paper to show that this property does indeed hold. To that end a preliminary study of the spin-$\frac{3}{2}$ equation is performed in detail by assuming a variable separation in each spinor field component. As expected, there results a set of separated angular equations that coincide with those of the R-W case (Zecca, 2006). Instead the radial separated dependence results in a set of ten coupled ordinary differential equations in the radial functions, whose degree of entanglement is greater than that relative to the R-W metric. By an elementary substitution method the scheme is reduced to the study of a system of four coupled ordinary differential equations that can be reported to independent equations in one only radial function. Even if asymptotic behaviours of the solutions can be determined, it remains open the problem of the exact solutions of the radial equations.

The result, namely the fact that the massive field equations of arbitrary spin are separable in the Schwarzschild metric follows by the recurrence structure of the equations. This structure, that is similar to that relative to the R-W metric, does hold for the spin-1 and spin-$\frac{3}{2}$ cases, and on account of the special values of the spin coefficients, is preserved by increasing the spin value. This suffices to show by induction the result.

2. SPIN-$\frac{3}{2}$ EQUATION IN SCHWARZSCHILD METRIC

The spin-$\frac{3}{2}$ field equation can be written in a general curved space-time as (Illge, 1993; Illge and Schimming, 1999)

$$\nabla^{D}_{X'}\phi_{DA_1A_2} + i\mu_\star\theta_{A_1A_2X'} = 0 \tag{1a}$$

$$\nabla^{'}_{(A}\theta_{A_1A_2)X'} - i\mu_\star\phi_{AA_1A_2} = 0 \tag{1b}$$

where $\mu_\star\sqrt{2}$ is the mass of the particle of the field and $\nabla_{AX'}$ the covariant spinor derivative. We look for solutions of Equation (1) under the assumptions $\phi_{ABC} = \phi_{(ABC)}$ and $\theta_{AB'} = \theta_{BA'}$ in the context of the Schwarzschild space-time of metric tensor

$$g_{\mu\nu} = \text{diag}\left\{1 - \frac{2M}{r}; \qquad -\left(1 - \frac{2M}{r}\right)^{-1}; \qquad -r^2; \qquad -r^2\sin^2\theta\right\} \tag{2}$$

To that end we develop the calculations in the Newman-Penrose formalism (Newman and Penrose, 1976) by choosing the null tetrad frame $\{l^i, n^i, m^i, m^{\star i}\}$ employed in Chandrasekhar's book (1983), whose corresponding directional derivatives and non trivial spinor coefficients, that we report fot reader's convenience, are

$$D = \partial_{00'} = l^i \partial_i = \frac{r}{r-2M}\partial_t + \partial_r, \tag{3}$$

$$\Delta = \partial_{11'} = n^i \partial_i = \frac{1}{2}\partial_t + \frac{2M-r}{2r}\partial_r \tag{4}$$

$$\delta = \partial_{01'} = m^i \partial_i = \frac{1}{r\sqrt{2}}(\partial_\theta + i \csc\theta\, \partial_\varphi), \tag{5}$$

$$\delta^\star = \partial_{10'} = m^{\star i} \partial_i = \frac{1}{r\sqrt{2}}(\partial_\theta - i \csc\theta\, \partial_\varphi). \tag{6}$$

$$\rho = -\frac{1}{r}, \qquad \mu = \frac{2M-r}{2r^2}, \quad \gamma = \frac{M}{2r^2}, \quad \beta = -\alpha = \frac{1}{2\sqrt{2}}\frac{\cot\theta}{r}. \tag{7}$$

By expliciting the Equation (1) in terms of the directional derivatives and spin coefficients (see e.g., Penrose and Rindler, 1984) one gets

$$(D - 3\rho)\phi_1 - (\delta^\star - 3\alpha)\phi_0 = i\mu_\star\theta_{000'} \tag{8}$$

$$(D - 2\rho)\phi_2 - (\delta^\star - \alpha)\phi_1 = i\mu_\star\theta_{010'} \tag{9}$$

$$(D - \rho)\phi_3 - (\delta^\star + \alpha)\phi_2 = i\mu_\star\theta_{110'} \tag{10}$$

$$(\Delta + \mu)\phi_0 - (\delta + \alpha)\phi_1 = -i\mu_\star\theta_{001'} \tag{11}$$

$$(\Delta + 2\mu)\phi_1 - (\delta + \alpha)\phi_2 = -i\mu_\star\theta_{101'} \tag{12}$$

$$(\Delta + 3\mu)\phi_2 - (\delta - 3\alpha)\phi_3 = -i\mu_\star\theta_{111'} \tag{13}$$

$$(D - \rho)\theta_{001'} - (\delta + \alpha)\theta_{000'} = -i\mu_\star\phi_0 \tag{14}$$

$$(D - \rho)\theta_{011'} - (\delta - \alpha)\theta_{010'} + \mu\theta_{000'} = -i\mu_\star\phi_1 \tag{15}$$

$$(D - \rho)\theta_{111'} - (\delta - 3\alpha)\theta_{110'} + 2\mu\theta_{010'} = -i\mu_\star\phi_2 \tag{16}$$

$$(\Delta + \mu)\theta_{000'} - (\delta^\star - 3\alpha)\theta_{001'} - 2\rho\theta_{101'} = i\mu_\star\phi_1 \tag{17}$$

$$(\Delta + \mu)\theta_{100'} - (\delta^\star - \alpha)\theta_{101'} - \rho\theta_{111'} = i\mu_\star\phi_2 \tag{18}$$

$$(\Delta + \mu)\theta_{110'} - (\delta^\star + \alpha)\theta_{111'} = i\mu_\star\phi_3 \tag{19}$$

It has been set $\phi_k \equiv \phi_{ABC} \;\Leftrightarrow\; k = A + B + C = 0, 1, 2, 3$. The second group of Equations (14)–(19) corresponds to the explicitation of equation (1b) without symmetrization. The equations have been maintained in this form to put in evidence the recurrence structure of the equation (as it happens also for spin-1 both in the present and in the R-W metric (Zecca, 2005, 2006), that will be usefull in the generalization of the result.

3. SEPARATION OF SPIN-$\frac{3}{2}$ EQUATION

To perform the calculations, it is better to consider the exact development of Equation (1b). Therefore, toghether with Equations (8)–(13) that are the explicitation of Equation (1a), one has to consider the four equations obtained from the symmetrization of Equations (14)–(19) that represent the exact explicitation of Equation (1b). These ten resulting equations can be separated by the positions

$$\phi_h = \varphi_h(r) S_h(\theta) F(t,\varphi), \quad F = e^{im\varphi + ikt}, \quad h = 0, 1, 2, 3, \tag{20}$$

$$\begin{aligned} \theta_{000'} &= F(t,\varphi) S_1(\theta) \theta_{00'}(r) & \theta_{001'} &= -F(t,\varphi) S_0(\theta) \theta_{01'}(r) \\ \theta_{010'} &= F(t,\varphi) S_2(\theta) \theta_{10'}(r) & \theta_{101'} &= -F(t,\varphi) S_1(\theta) \theta_{11'}(r) \\ \theta_{110'} &= F(t,\varphi) S_3(\theta) \theta_{20'}(r) & \theta_{111'} &= -F(t,\varphi) S_2(\theta) \theta_{21'}(r) \end{aligned} \tag{21}$$

We assume $m = 0, \pm 1, \pm 2, \ldots$. The resulting separated angular equations are

$$\begin{array}{lll} L^-_{\frac{3}{2}} S_0 = \lambda_1 S_1 & L^-_{\frac{1}{2}} S_1 = \lambda_2 S_2 & L^-_{-\frac{1}{2}} S_2 = \lambda_3 S_3 \\ L^+_{-\frac{1}{2}} S_1 = \lambda_4 S_0 & L^+_{\frac{1}{2}} S_2 = \lambda_5 S_1 & L^+_{\frac{3}{2}} S_3 = \lambda_6 S_2 \end{array} \tag{22}$$

$\lambda_i,\ i = 1, 2, \ldots, 6$ separation constants and it has been set $L^{\pm}_d = \partial_\theta \mp m \csc\theta + d\cot\theta$. These equations are the same as those relative to the spin-$\frac{3}{2}$ in R-W space-time. By setting $\lambda_1\lambda_4 = \lambda_3\lambda_6 = \lambda_2\lambda_5 + 1 = \lambda^2$ the Equation (22) can be consistently put into the form of an eigenvalue problem that can be analytically solved (see Zecca, 2006). The result is that λ^2 takes the values $\lambda^2 + \frac{33}{8} = l(l+1),\ \ l = 2, 3, \ldots; S_1,\ S_2$ are the same as those of the spin-$\frac{1}{2}$ case, while the function S_0, that can be obtained from S_3 by the substitution $m \to -m$, is essentially expressed in terms of Jacobi polynomials.

By using the positions (20), (21) into the mentioned symmetrized equations, the separated radial equations, that we write in a compact form, result to be

$$A_3\phi_1 - \frac{\lambda_1}{r\sqrt{2}}\phi_0 = i\mu_\star \theta_{00'} \tag{23}$$

$$A_2\phi_2 - \frac{\lambda_2}{r\sqrt{2}}\phi_1 = i\mu_\star \theta_{10'} \tag{24}$$

$$A_1\phi_3 - \frac{\lambda_3}{r\sqrt{2}}\phi_2 = i\mu_\star \theta_{20'} \tag{25}$$

$$B_0\phi_0 - \frac{\lambda_4}{r\sqrt{2}}\phi_1 = i\mu_\star \theta_{01'} \tag{26}$$

$$B_1\phi_1 - \frac{\lambda_5}{r\sqrt{2}}\phi_2 = i\mu_\star\theta_{11'} \tag{27}$$

$$B_2\phi_2 - \frac{\lambda_6}{r\sqrt{2}}\phi_3 = i\mu_\star\theta_{21'} \tag{28}$$

$$A_1\theta_{01'} + \frac{\lambda_4}{r\sqrt{2}}\theta_{00'} = i\mu_\star\phi_0 \tag{29}$$

$$2A_0\theta_{11'} + \left(B_{-1} + \frac{1}{2r}\right)\theta_{00'} + \frac{1}{r\sqrt{2}}(2\lambda_5\theta_{10'} + \lambda_1\theta_{01'}) = 3i\mu_\star\phi_1 \tag{30}$$

$$A_{-1}\theta_{21'} + \left(2B_{-1} + \frac{4M}{r^2}\right)\theta_{10'} + \frac{1}{r\sqrt{2}}(2\lambda_6\theta_{20'} + 2\lambda_2\theta_{11'}) = 3i\mu_\star\phi_2 \tag{31}$$

$$\left(B_1 + \frac{1}{2r}\right)\theta_{20'} + \frac{\lambda_3}{r\sqrt{2}}\theta_{21'} = i\mu_\star\phi_3 \tag{32}$$

where it has been defined

$$A_d = \frac{d}{dr} + \frac{ikr}{r-2M} + \frac{d}{r}$$
$$B_f = \frac{2M-r}{2r}\frac{d}{dr} + \frac{ik}{2} + \frac{M(4f+1) - r(f+1)}{2r^2}. \tag{33}$$

By using the Equations (23)–(28) into Equations (29)–(32) the solution of the system of equations can be reduced to the solution of the following coupled differential equations

$$\frac{3\sqrt{2}\lambda_2 r\phi_1}{r-2M} = r^2\phi_0'' + r\frac{M+2r}{r-2M}\phi_0' - \left[\frac{M + r\lambda^2 + 2\mu_\star^2 r^3}{r-2M} - \frac{k^2r^4 + 3ikMr^2}{(r-2M)^2}\right]\phi_0 \tag{34}$$

$$\frac{6\lambda_5\phi_2}{r^2\sqrt{2}} - \frac{3\lambda_1(r-2M)}{2r^3\sqrt{2}}\phi_o = \left[2A_0B_1 + B_{-1}A_3 + \frac{M}{2r}A_3 + \frac{3\lambda^2+2}{2r^2} + 3\mu_\star^2\right]\phi_1 \tag{35}$$

$$\frac{3\lambda_2(2M-r)\phi_1}{r^3\sqrt{2}} + \frac{3\lambda_6\phi_3}{r^2\sqrt{2}} = \left[A_{-1}B_2 + 2B_{-1}A_2 + \frac{4M}{r^3}A_2 + \frac{3\lambda^2+2}{2r^2} + 3\mu_\star^2\right]\phi_2 \tag{36}$$

$$\frac{3\lambda_3}{\sqrt{2}}\phi_2 = -r^2\phi_3'' - \frac{r(2r-5M)}{r-2M}\phi_3' + \left(\frac{M + r\lambda^2 + 2\mu_\star^2 r^3}{r-2M} - \frac{k^2r^4 - 3ikMr^2}{(r-2M)^2}\right)\phi_3 \tag{37}$$

$\lambda^2 = l(l+1) - 33/8, l = 2, 3\cdots$. In turns these equations can be given the form of equations in one only unknown function. As an example, from Equations (34) and (35) one has $\phi_2 = F_2\phi_0$ that toghether with (36), (34) implies $\phi_3 = F_3\phi_0$. On the other hand from (35), (36) ϕ_0 can be expressed as a function of ϕ_2, ϕ_3 and hence $\phi_0 = F_0\phi_3$ by (37). Therefore $\phi_3 = F_3F_0\phi_3$ and $\phi_0 = F_0F_3\phi_0$, F_0, F_3 linear differential operators of higher order. It is evident that the explicit solution of these last equations remains difficult. It is however easy to see that the Equations (34)–(37) admit asymptotic solutions of the form

$$\begin{aligned} \phi_1 \cong \phi_2 &\quad \stackrel{r\to\infty}{\longrightarrow} \quad e^{\pm i\omega r}, \\ \phi_0 \cong \phi_3 &\quad \stackrel{r\to\infty}{\longrightarrow} \quad \frac{e^{\pm i\omega r}}{r}. \end{aligned} \tag{38}$$

where $\omega = \sqrt{k^2 - 2\mu_\star^2}$.

4. FIELD EQUATION OF ARBITRARY SPIN

The massive field equations for spin $s = \frac{n+1}{2}$ can be consistently formulated in curved space-time (see Illge, 1993 and Illge and Schimming, 1999) as

$$\nabla_{\dot{X}}^{D}\,\phi_{DA_1A_2\ldots A_n} = -i\mu_\star\,\theta_{A_1A_2\ldots A_n'} \tag{39a}$$

$$\nabla'_{(A}\,\theta_{A_1A_2\ldots A_n)X'} = i\mu_\star\phi_{AA_1A_2\ldots A_n} \tag{39b}$$

where again $\mu_\star\sqrt{2}$ is the mass of the particles of the field and the spinors are assumed to be symmetric in the unprimed indexes: $\phi_{DA_1A_2\ldots A_n} = \phi_{(DA_1A_2\ldots A_n)}$ and $\theta_{A_1A_2\ldots A_nX'} = \theta_{(A_1A_2\ldots A_n)X'}$. To show that the Equation (39) are separable for arbitrary spin one can proceed, a fortiori, as for the case of the R-W space-time because here also the spin coefficients ϵ vanishes. The result follows by remarking that passing from order $n-1$ to order n amounts in adding to the detailed expressions of the spinor derivatives in Equations (39a) and (39b) respectively the terms

$$\phi^{D}_{\ A_1A_2\ldots A_{n-1}X}\ \Gamma^{X}_{DX'A_n}$$

$$\theta_{(A_1A_2\ldots A_{n-1}X)X'}\,\Gamma^{X}_{AX'A_n}$$

By expliciting $\Gamma_{D'A_n}{}^{X}$ in terms of the spin coefficients (Penrose and Rindler, 1984) one can check (e.g. Zecca, 1996b) that by keeping $A_1A_2\ldots A_{n-1}$ fixed and letting $h = A_1 + A_1 + \ldots + A_{n-1}$, the term (40a) introduces in the equation of order $n-1$ only terms ϕ_h, ϕ_{h+1} for $A_n = 0$ and ϕ_{h+1}, ϕ_{h+2} for $A_n = 1$. Therefore by induction over n the structure of the left hand side of Equation (39a) contains always two ϕ components as in Equations (8)–(13) for $s = \frac{3}{2}$. Increasing n increases the number of times the spin coefficients appear in the equation.

There follows that the obvious generalization of the assumption (20), (21) separates not only Equation (39a) but also Equation (39b), by induction, because this is true for $s = 1$ [Zecca, 2005] and $s = \frac{3}{2}$ (present work). Indeed, by an argument like to the above, the term (40b) maintains in two or three the number of the θ components involved in the explicitation of the left hand side of Equation (39b), whose structure is the obvious generalization by recurrence of Equations (15)–(19).

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